

# DISTURBING THE BLACK HOLE

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**Abstract.** I describe some examples in support of the conjecture that the horizon area of a near equilibrium black hole is an adiabatic invariant. These include a Schwarzschild black hole perturbed by quasistatic scalar fields (which may be minimally or nonminimally coupled to curvature), a Kerr black under the influence of scalar radiation at the superradiance threshold, and a Reissner–Nordström black hole absorbing a charge marginally. These clarify somewhat the conditions under which the conjecture would be true. The desired “adiabatic theorem” provides an important motivation for a scheme for black hole quantization.

## 1. Introduction

Does the event horizon area of a black hole always grow under external perturbations ? Hawking’s area theorem [2] would suggest an affirmative answer whenever classical fields obeying the weak energy condition are involved. Nevertheless, one can categorize a variety of situations in which an external perturbation transmitted through common fields is slowly applied and relaxed, and does *not* lead to area increase. This classical “adiabatic invariance” of horizon area, not yet a theorem but a collection of examples, is obviously consistent with the entropy character of black hole area [3, 4] because in classical thermodynamics entropy is invariant under slow changes of an insulated system in thermodynamic equilibrium. It is also an important motivation in an approach to black hole quantization [5, 6, 7, 8] which has received increasing attention in the last couple of years. [9] Keeping the ultimate application of the adiabatic property to black hole quantization in the back of the mind will help in grasping the significance of the medley of examples here garnered. In collecting these I have had in mind generating interest in turning the observation into a precise theorem.

Consider a small patch of event horizon area  $\delta A$ ; it is formed by null generators whose tangents are  $l^\alpha = dx^\alpha/d\lambda$ , where  $\lambda$  is an affine parameter along the generators. By definition of the convergence  $\rho$  of the generators,  $\delta A$  changes at a rate

$$d\delta A/d\lambda = -2\rho\delta A. \quad (1)$$

Now  $\rho$  itself changes at a rate given by the optical analogue of the Raychaudhuri equation (with Einstein's equations already incorporated; I use units such that  $G = c = 1$ ) [10, 11]

$$d\rho/d\lambda = \rho^2 + |\sigma|^2 + 4\pi T_{\alpha\beta} l^\alpha l^\beta, \quad (2)$$

where  $\sigma$  is the shear of the generators and  $T_{\alpha\beta}$  the energy momentum tensor. The shear evolves according to

$$d\sigma/d\lambda = 2\rho\sigma + (3\epsilon - \bar{\epsilon})\sigma + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \quad (3)$$

where  $C_{\alpha\beta\gamma\delta}$  is the Weyl conformal tensor,  $m^\alpha$  one of the Newman–Penrose tetrad legs which lies in the horizon, and  $\epsilon$  a pure imaginary parameter.

Many types of classical matter obey the weak energy condition

$$T_{\alpha\beta} l^\alpha l^\beta \geq 0. \quad (4)$$

Whenever this is true,  $\rho$  can - according to Eq. (2) - only *grow* or remain unchanged along the generators. Now were  $\rho$  to become positive at any event along a generator of our horizon patch, then by Eq. (2) it would remain positive henceforth, and indeed grow bigger. The joint solution of Eqs. (1)-(2) shows that  $\delta A$  would shrink to nought in a finite span of  $\lambda$ . [2, 12] This vanishing with its implied extinction of generators would constitute a singularity on the horizon. But it is an axiom of the subject [2] that event horizon generators cannot end in the future. The only way out is to accept that  $\rho \leq 0$  everywhere along the generators, which by Eq. (1) signifies that the horizon patch's area can never decrease. This is the essence of Hawking's area theorem.

It will be noticed that to keep the horizon area constant requires  $\rho = 0$  which by Eqs. (2)-(3) implies that both  $C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta$  and  $T_{\alpha\beta} l^\alpha l^\beta$  vanish at the horizon. Vanishing of  $C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta$  requires that the geometry be quasistationary to prevent gravitational waves, which are quantified by  $C_{\alpha\beta\gamma\delta}$ , from impinging on the horizon. Thus with a quasistationary geometry, preservation of the horizon's area requires

$$T_{\alpha\beta} l^\alpha l^\beta = 0 \quad \text{on the horizon.} \quad (5)$$

Contrary to the folklore which considers increase of horizon area to be an almost compulsory consequence of changes in the black hole, I shall here

exhibit a variety of situations for which the conditions that keep horizon area unchanged occur naturally. The rule that seems to emerge is that quasistationary changes of the black hole occasioned by an external influence will leave the horizon area unchanged. This means an “adiabatic theorem” for black holes must exist.

## 2. Black Hole Disturbed by Scalar Charges

Consider a Schwarzschild black hole with exterior metric

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (6)$$

Suppose sources of a *minimally coupled* scalar field  $\Phi$  are brought up slowly from infinity to a finite distance of the hole, and then withdrawn equally slowly. Does this changing influence increase the horizon’s area? Given that the changes are quasistatic, the question is just whether condition (5) is satisfied for all time. As we shall point out in Sec. 3, a potential barrier at  $r \sim 3M$  screens out quasistatic scalar perturbations. For this reason our analysis becomes more than academic only when the sources actually enter the region  $2M < r < 3M$ .

If the scalar’s sources are weak, one may regard  $\Phi$  as a quantity of first order, and proceed by perturbation theory. The scalar’s energy–momentum tensor,

$$T_\alpha{}^\beta = \nabla_\alpha \Phi \nabla^\beta \Phi - \frac{1}{2} \delta_\alpha^\beta \nabla_\gamma \Phi \nabla^\gamma \Phi, \quad (7)$$

will be of second order of smallness. I shall suppose the same is true of the energy–momentum tensor of the sources themselves. Thus to first order the metric (6) is unchanged. Neglecting for the moment time derivatives, the scalar equation outside the scalar’s sources can be written in the form

$$\partial/\partial r \left[ (r^2 - 2Mr) \partial\Phi/\partial r \right] - \hat{L}^2 \Phi = 0, \quad (8)$$

where  $\hat{L}^2$  is the usual squared angular momentum operator (but without the  $\hbar^2$  factor). This equation suggests looking for a solution of the form

$$\Phi = \Re \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}(r) Y_{\ell m}(\theta, \varphi), \quad (9)$$

where the  $Y_{\ell m}$  are the familiar spherical harmonic (complex) functions. Since the  $Y_{\ell m}$  form a complete set in angular space, any function  $\Phi(r, \theta, \varphi)$  can be so expressed. And since  $\hat{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$ , the radial and angular variables separate, and one finds for  $f_{\ell m}$  the equation

$$d/dr \left[ (r^2 - 2Mr) df_{\ell m}/dr \right] - \ell(\ell+1)f_\ell = 0. \quad (10)$$

Since the index  $m$  does not figure in this equation, I write just plain  $f_\ell(r)$ ; one may obviously pick  $f_\ell(r)$  to be real.

Let us change from variable  $r$  to  $x \equiv r/M - 1$  and define  $F_\ell(x) \equiv f_\ell(r)$ , so that Eq. (10) becomes

$$d/dx \left[ (1 - x^2) F_\ell \right] + \ell(\ell + 1) F_\ell = 0. \quad (11)$$

This is the Legendre equation of order  $\ell$ . Its solutions regular at the singular point  $x = 1$  of the equation are the well known Legendre polynomials  $P_\ell(x)$ . Independent solutions are furnished by the Legendre associated functions  $Q_\ell$  which have the general form [13]

$$Q_\ell(x) = \frac{1}{2} \ln \left[ \frac{x+1}{x-1} \right] P_\ell(x) + \text{polynomial of order } (\ell - 1) \text{ in } x. \quad (12)$$

The associated solutions are thus singular at the horizon  $x = 1$ , and must not be included in  $\Phi$ , as the following argument makes clear.

We obviously require that the event horizon remain regular under the scalar's perturbation; otherwise the black hole is destroyed and our discussion is over before it began. A minimal requirement for regularity is that physical invariants like  $\Upsilon_1 \equiv T_\alpha^\alpha$ ,  $\Upsilon_2 \equiv T_\alpha^\beta T_\beta^\alpha$ ,  $\Upsilon_3 \equiv T_\alpha^\beta T_\beta^\gamma T_\gamma^\alpha$ , *etc.*, be bounded, for divergence of any of them would surely induce curvature singularities via the Einstein equations. By Eq. (7) the invariant  $\Upsilon_k$  is always proportional to  $(\Phi_{,\alpha} \Phi^{,\alpha})^k$ . Now to lowest order of smallness the metric components that enter into  $\Upsilon_k$  are just the Schwarzschild ones. In particular, in our static case  $\Phi_{,\alpha} \Phi^{,\alpha} = (1 - 2M/r) \Phi_{,r}^2 + \dots$ . One must thus require

$$(1 - 2M/r)^{1/2} \Phi_{,r} \text{ bounded at horizon.} \quad (13)$$

Since this condition must hold for every  $\theta$  and  $\varphi$ , it follows from the independence of the various  $Y_{\ell m}$  in Eq. (9) that

$$\forall \ell: \quad (1 - 2M/r)^{1/2} df_\ell/dr \text{ bounded.} \quad (14)$$

But according to Eq. (12) the  $Q_\ell$  are too singular to satisfy this equation, *i.e.*,  $\sqrt{x-1} Q'_\ell(x) \rightarrow \infty$  as  $x \rightarrow 1$ . Thus one must discard the  $Q_\ell$  from the set of radial solutions relevant in the region between the innermost source and the horizon.

Thus in this inner region of the black hole exterior we have, to first order in perturbation theory, the exact solution

$$\Phi = \Re \left( \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} P_\ell(r) Y_{\ell m}(\theta, \varphi) \right), \quad (15)$$

where the (complex) coefficients  $C_{\ell m}$  permit us to match the solution to every distribution of sources by the usual methods. As those sources are moved around slowly, the  $C_{\ell m}$  will change slowly (we shall investigate the question of changes at finite speed in Sec. 3). Now according to Eq. (7),

$$T_r{}^r - T_t{}^t = (1 - 2M/r) \Phi_{,r}^2. \quad (16)$$

Because  $\Phi$  is completely regular down to the horizon, this shows that

$$\lim_{r \rightarrow 2M} (T_r{}^r - T_t{}^t) = 0. \quad (17)$$

We now explain the significance of this general result [14, 15] for our specific problem.

Any 3D-hypersurface of the form  $\{\forall t, r = \text{const.}\}$  has a tangent  $\tau^\alpha = \delta_t^\alpha$  with norm  $-(1 - 2M/r)$  as well as the normal  $\eta_\alpha = \partial_\alpha(r - \text{const.}) = \delta_\alpha^r$  with norm  $(1 - 2M/r)$ . The vector  $N^\alpha \equiv \tau^\alpha + (1 - 2M/r)\eta^\alpha$  is obviously null, and as  $r \rightarrow 2M$ , both its covariant and contravariant forms remain well defined. Indeed, as  $r \rightarrow 2M$  the other two vectors become null; in contrast with them,  $N^\alpha$  remains well behaved at the horizon, so that it must there be proportional (with finite nonvanishing proportionality constant) to  $l^\alpha$ , the tangent to the horizon generator. This can be verified by remarking that  $N^\alpha$ , just as  $l^\alpha$ , is future pointed ( $N^t > 0$ ) as well as outgoing ( $N^r > 0$ ). Now at any point  $r \geq 2M$

$$T_{\alpha\beta} N^\alpha N^\beta = T_t{}^t N_t N^t + T_r{}^r N_r N^r = (T_r{}^r - T_t{}^t) N_r N^r. \quad (18)$$

But  $N_r N^r \rightarrow 0$  at the horizon, so that in view of Eq. (17),  $T_{\alpha\beta} l^\alpha l^\beta = 0$ .

Since Eq. (5) is satisfied, the horizon's area does not change under the action of the scalar field. This conclusion is obviously conditional on the changes of the scalar's sources taking place sufficiently slowly, for otherwise Eq. (8) would be invalid, and the components of  $T_\mu{}^\nu$  I employed would be affected. Our result supports the notion that adiabatic perturbations of a Schwarzschild black hole do not change the area, so that horizon area is an adiabatic invariant.

Notice that  $T_{\alpha\beta} l^\alpha l^\beta$  vanishes because it is a product of *two* vanishing factors. This suggests that the area invariance result will remain valid under small "perturbations" to our scenario. Let us investigate the issue of time dependence of the scalar field; it is important because I am contemplating moving the sources of the scalar field so that it is never perfectly static as assumed heretofore.

### 3. The Time Dependent Problem

Let us retain in the scalar equation the time derivatives:

$$-\frac{r^4}{(r^2 - 2Mr)} \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial r} \left[ (r^2 - 2M) \frac{\partial \Phi}{\partial r} \right] - \hat{L}^2 \Phi = 0. \quad (19)$$

In analogy with Eq. (15) we now look for a solution of the form

$$\Phi = \Re \int_0^\infty d\omega \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell C_{\ell m}(\omega) f_\ell(\omega, r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}. \quad (20)$$

In terms of Wheeler's "tortoise" coordinate  $r^* \equiv r + 2M \ln(r/2M - 1)$ , for which the horizon resides at  $r^* = -\infty$ , the equation satisfied by the new radial function  $H_{\ell\omega}(r^*) \equiv r f_\ell(\omega, r)$  is [12]

$$-\frac{d^2 H_{\ell\omega}}{dr^{*2}} + \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right) H_{\ell\omega} = \omega^2 H_{\ell\omega}. \quad (21)$$

The analogy between Eq. (21) and the Schrödinger eigenvalue equation permits the following analysis [12] of the effects of distant scalar sources on the black hole horizon. Waves with "energy"  $\omega^2$  on their way in from a distant source run into a positive potential, the product of the two parentheses in Eq. (21). The potential's peak is situated at  $r \approx 3M$  for all  $\ell$ . Its height is  $0.0264M^{-2}$  for  $\ell = 0$ ,  $0.0993M^{-2}$  for  $\ell = 1$  and  $\approx 0.038 \ell(\ell+1)M^{-2}$  for  $\ell \geq 2$ . Therefore, waves with any  $\ell$  and  $\omega < 0.163M^{-1}$  coming from sources at  $r \gg 3M$  have to tunnel through the potential barrier to get near the horizon. As a consequence, the wave amplitudes that penetrate to the horizon are small fractions of the initial amplitudes, most of the waves being reflected back. In fact, the tunnelling coefficient vanishes in the limit  $\omega \rightarrow 0$ . [12] This means that adiabatic perturbations by distant sources (which surely means they only contain Fourier components with  $\omega \ll M^{-1}$ ) perturb the horizon very weakly (this is just an inverse of Price's theorem [12] that a totally collapsed star's asymptotic geometry preserves no memories of the star's shape). Thus one would not expect significant growth of horizon area from scalar perturbations originating in distant sources.

What if the scalar's sources are moved into the region  $2M < r < 3M$  inside the barrier? They will now be able to perturb the horizon; do they change it's area? To check I look for the solutions of Eq. (21) in the region near the horizon where the potential is small compared to  $\omega^2$ ; according to the theory of linear second order differential equations they are of the form

$$H_{\ell\omega}(r^*) = \exp(\pm i\omega r^*) \times [1 + O(1 - 2m/r)]. \quad (22)$$

The Matzner boundary condition [16] that the physical solution be an incoming wave as appropriate to the absorbing character of the horizon selects the sign in the exponent as negative. Hence the typical term in  $\Phi$  is

$$\frac{1 + O(1 - 2m/r)}{r} P_\ell(\cos \theta) \cos \psi; \quad \psi \equiv \omega(r^* + t) - m\varphi. \quad (23)$$

Is the perturbation well behaved at the horizon? In particular, are all the invariants  $\Upsilon_k$  of Sec. 2 (or equivalently  $\Phi_{,\alpha} \Phi^{,\alpha}$ ) bounded there? Let us first look at a  $\Phi$  composed of a single mode like that in Eq. (23). An explicit calculation on the Schwarzschild background using  $dr^*/dr = (1 - 2M/r)^{-1}$  gives, after a miraculous cancellation of terms divergent at the horizon (pointed out by A. Mayo),

$$\Phi_{,\alpha} \Phi^{,\alpha} \propto \frac{m^2 P_\ell^2 \sin^2 \psi}{r^4 \sin^2 \theta} + \left( \frac{dP_\ell}{d\theta} \right)^2 \frac{\cos^2 \psi}{r^4} + \frac{\omega \sin(2\psi)}{r^3} P_\ell^2 + \dots, \quad (24)$$

where “ $\dots$ ” here and henceforth denote terms that vanish as  $r \rightarrow 2M$ . This expression is bounded at the horizon. Now suppose  $\Phi$  is the sum of two modes like (23). Let us label the various parameters with subscripts “1” and “2”. Then a calculation gives  $\Phi_{,\alpha} \Phi^{,\alpha}$  as consisting of three groups of terms, two of them of form (24) with subscripts 1 and 2, respectively, and a third of the form

$$\begin{aligned} & \frac{m_1 m_2 P_{\ell_1} P_{\ell_2} \sin \psi_1 \sin \psi_2}{r^4 \sin^2 \theta} + \left( \frac{dP_{\ell_1}}{d\theta} \right) \left( \frac{dP_{\ell_2}}{d\theta} \right) \frac{\cos \psi_1 \cos \psi_2}{r^4} + \\ & + \frac{\omega_1 \sin \psi_1 \cos \psi_2 + \omega_2 \sin \psi_2 \cos \psi_1}{r^3} P_{\ell_1} P_{\ell_2} + \dots \end{aligned} \quad (25)$$

This is also bounded. By induction any  $\Phi$  of form (20) will give a bounded  $\Phi_{,\alpha} \Phi^{,\alpha}$ . Thus all the  $\Upsilon_k$  are bounded at  $r = 2M$ , and a generic scalar perturbation does not disturb the horizon unduly.

The extent by which the *shape* of the horizon is perturbed must be linear in the magnitude of the invariant  $\Upsilon_1$  (Einstein’s equations have  $T_{\alpha\beta}$  as source, not  $T_\alpha{}^\gamma T_\gamma{}^\beta$ ). It is then clear from both our results that this perturbation is of order  $O(\omega^0)$  generically, and of  $O(\omega)$  in the monopole case. I now show that the change in the horizon area is of  $O(\omega^2)$ , so that for small  $\omega$  the area is (relatively) invariant.

Because now there is time variation and  $T_t{}^r \neq 0$ , Eq. (18) has to be generalized to the form

$$T_{\alpha\beta} N^\alpha N^\beta = (T_r{}^r - T_t{}^r) N_r N^r + 2T_t{}^r N_r N^t. \quad (26)$$

From  $N^\alpha$ ’s definition in Sec. 2 we have  $N_r N^r = 1 - 2M/r$  and  $N_r N^t = 1$ . And from Eq. (7) it is clear that  $T_r{}^r - T_t{}^t = \Phi_{,r} \Phi^{,r} - \Phi_{,t} \Phi^{,t}$  while  $T_t{}^r = \Phi_{,t} \Phi^{,r}$ . Thus

$$T_{\alpha\beta} N^\alpha N^\beta = [\Phi_{,t} + (1 - 2M/r) \Phi_{,r}]^2. \quad (27)$$

If one now substitutes a  $\Phi$  made up of a single mode like in Eq. (23), one concludes that

$$T_{\alpha\beta}l^\alpha l^\beta \propto \frac{\omega^2 P_\ell^2 \sin^2 \psi}{r^2} + \dots. \quad (28)$$

A quick way to this result is to recognize that  $l^\alpha \propto \tau^\alpha \equiv (\partial/\partial t)^\alpha$  because the horizon generators must lie along the only Killing vector of the problem. In view of Eq. (7) and the null character of  $l^\alpha$ ,

$$T_{\alpha\beta}l^\alpha l^\beta \propto (\Phi_{,\alpha} \tau^\alpha)^2 = (\partial\Phi/\partial t)^2, \quad (29)$$

which reproduces Eq. (28). And if one substitutes the generic  $\Phi$ , the proportionality to the square of frequency will obviously remain.

Thus, when scalar field sources are moved inside the barrier, they perturb the geometry, and the horizon's shape in particular, by an amount which does not, in general, vanish as the perturbations change very slowly. By contrast, the rate of change of the horizon area vanishes as the square of the typical Fourier frequency of the perturbation. In this sense the horizon area is an adiabatic invariant.

#### 4. Generalization to Nonminimally Coupled Field

To what extent is our result generic? For example, does it depend on the nature of the interaction? One can probe this question by replacing our minimally coupled scalar field by one coupled nonminimally to curvature. The scalar equation is replaced by [17]

$$(\nabla_\alpha \nabla^\alpha - \xi R)\Phi = 0, \quad (30)$$

where  $R$  denotes the Ricci scalar and  $\xi \neq 0$  is a real constant measuring the extent of nonminimal coupling;  $\xi = 1/6$  corresponds to a conformally invariant scalar equation. The corresponding energy-momentum tensor is [18]

$$T_\alpha{}^\beta = \nabla_\alpha \Phi \nabla^\beta \Phi - \frac{1}{2} \nabla_\alpha \Phi \nabla^\alpha \Phi \delta_\alpha^\beta - \xi (\nabla_\alpha \nabla^\beta \Phi^2 - \delta_\alpha^\beta \nabla_\mu \nabla^\mu \Phi^2 - G_\alpha{}^\beta \Phi^2), \quad (31)$$

where  $G_\alpha{}^\beta$  denotes the Einstein tensor. Evaluating it with help of the Einstein equations one obtains, in the region outside the scalar's sources where they do not contribute to  $T_\alpha{}^\beta$ ,

$$T_\alpha{}^\beta = \frac{\nabla_\alpha \Phi \nabla^\beta \Phi - \frac{1}{2} \delta_\alpha^\beta \nabla_\gamma \Phi \nabla^\gamma \Phi - \xi (\nabla_\alpha \nabla^\beta \Phi^2 - \delta_\alpha^\beta \nabla_\gamma \nabla^\gamma \Phi^2)}{1 - 8\pi G \xi \Phi^2}. \quad (32)$$

The following analysis is carried out with neglect of time variation as in Sec. 2. If again one regards  $\Phi$  as of first order of smallness, then it is



clear that  $T_\alpha^\beta$  is of second order and consequently the lowest correction to the Schwarzschild metric Eq. (6) is of second order. Likewise,  $R$  in Eq. (30) is of second order. Therefore, in the static case one may again get  $\Phi$  to first order by solving Eq. (8), and  $\Phi$  is given by the series in Eq. (9) with radial functions which are superpositions of Legendre polynomials  $P_\ell$  and Legendre functions  $Q_\ell$ .

To select out the physical combinations of these, I again require - the argument is exactly like in Sec. 2 - that every *diagonal* component of  $T_\alpha^\beta$  be bounded. In particular one has from Eq. (32) that

$$T_t^t - T_\varphi^\varphi = \frac{2\xi(1 - 3M/r)\Phi\Phi_{,r}}{r(1 - 8\pi G\xi\Phi^2)} \quad \text{bounded at horizon.} \quad (33)$$

It is clear from this that  $\Phi$  itself must be bounded at  $r = 2M$ , for if it were to diverge there, then  $|T_t^t - T_\varphi^\varphi| \sim |d \ln \Phi / dr| \rightarrow \infty$ . I thus discard all the  $Q_\ell$  functions from  $\Phi$ , thereby returning to the form (15):  $\Phi$  is regular at the horizon.

Now

$$T_t^t - T_r^r = \left(1 - \frac{2M}{r}\right) \frac{(2\xi - 1)\Phi_{,r}^2 + 2\xi\Phi\Phi_{,rr}}{1 - 8\pi G\xi\Phi^2}. \quad (34)$$

From this it is clear that  $T_t^t - T_r^r \rightarrow 0$  at  $r = 2M$  unless  $8\pi G\xi\Phi^2 \rightarrow 1$  there. This last possibility must be excluded for it would be equivalent to having  $\ln(1 - 8\pi G\xi\Phi^2) \rightarrow -\infty$ , whereby  $d \ln(1 - 8\pi G\xi\Phi^2)/dr$  would necessarily diverge at  $r = 2M$ . But this derivative occurs as a factor in  $T_t^t - T_\varphi^\varphi$  and its divergence would not be compensated for; thus Eq. (33) would be contradicted. Since  $T_t^t - T_r^r \rightarrow 0$  on the horizon, the arguments at the end of Sec. 2 can be repeated to show that the area of the black hole is unchanged by the scalar perturbation. Although we shall not go here into the time dependent problem, it seems that coupling the field nonminimally makes little difference regarding the adiabatic invariance of the horizon.

A. Mayo [19] has worked out the effect of static electromagnetic perturbations on the horizon and shown that they also leave its area unchanged.

## 5. Waves Impinging on a Kerr Black Hole

In the above two examples when the perturbation is removed, the black hole exterior must return to exact Schwarzschild form because a static spherical black hole with no electric charge cannot retain scalar hair [20]. Further, the black hole returns to its original mass since, for the Schwarzschild case, the horizon area and the mass are related one-to-one, and the area has not changed. The question thus arises, does adiabatic invariance of the area continue to hold when the black hole's perturbation is accompanied by a net change in some of the Wheeler black hole parameters mass, charge and

angular momentum ? We now show the answer is affirmative for adiabatic scalar perturbations of the Kerr black hole.

In the Kerr case the meaning of “adiabatic” needs to be refined. It is well known that a static ( $\omega = 0$ ) but nonaxisymmetric perturbation of a Kerr black hole, such as would be caused by field sources held in its vicinity at rest with respect to infinity, necessarily causes an increase in horizon area. [21] However, static perturbations in this sense are not adiabatic from the local point of view. Because of the dragging of inertial frames, [12] any nonaxisymmetric static field is perceived by momentarily radially stationary local inertial observers as endowed with temporal variation as these observers are necessarily dragged through the field’s spatial inhomogeneity. At the horizon the dragging frequency is the hole’s rotational frequency  $\Omega$ , and a field component with azimuthal “quantum” number  $m$  is seen to vary with temporal frequency  $m\Omega$  which need not be small. Evidently, “adiabatic” must here mean that *according to momentarily radially stationary local inertial observers*, the perturbation has only low frequency Fourier components.

To see these concepts in practice, consider a Kerr black hole of mass  $M$  and angular momentum  $J$ . I shall not need the metric; all that is important here is that the horizon area is

$$A = 4\pi \left[ \left( M + \sqrt{M^2 - (J/M)^2} \right)^2 + (J/M)^2 \right]. \quad (35)$$

Let sources distributed well away from the hole radiate upon it a *weak* scalar wave of the form

$$\Phi = f(r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}. \quad (36)$$

In the spirit of perturbation theory I shall neglect the gravitational waves so produced. The black hole geometry will eventually be changed by interaction with this wave, but since the latter is taken to be weak, I shall assume that the change amounts to a transition from one Kerr geometry to another with slightly different  $M$  and  $J$ . In the final analysis such assumption is justified by the stability of the Kerr geometry. Since the geometry thus remains axisymmetric and stationary after the change, the wave preserves its form (36) over all time. Long ago Starobinskii [22] showed that for small  $\omega - m\Omega$  the absorption coefficient for a wave like (36) has the form

$$\Gamma = K_{\omega \ell m}(M, J) \cdot (\omega - \Omega m), \quad (37)$$

where

$$\Omega \equiv \frac{J/M}{r_{\mathcal{H}}^2 + (J/M)^2}; \quad \text{with} \quad r_{\mathcal{H}} \equiv M + \sqrt{M^2 - (J/M)^2} \quad (38)$$

is the rotational angular frequency of the hole, while  $K_{\omega\ell m}(M, J)$  is a positive coefficient. If one chooses  $\omega = m\Omega$  (static perturbation in the eyes of the local inertial observers), the wave is perfectly reflected and no change of the black hole parameters ensues.

By choosing  $\omega - \Omega m$  slightly positive, one arrange for a small fraction of the wave to get absorbed. Now imagine surrounding the system by a large spherical mirror. The part of the wave reflected off the black hole gets reflected perfectly back towards it by the mirror. The wave thus bounces back and forth between the two and a sizable fraction of its energy and angular momentum will eventually get absorbed by the hole. Similarly, if one makes  $\omega - m\Omega$  slightly negative, the wave gets amplified upon reflection off the hole (Zel'dovich–Misner superradiance [23]) and the repeated reflections make it stronger than originally. Since both processes envisioned take place over many cycles of reflection (a long interval of  $t$  time), the consequent (substantial) changes of  $M$  and  $J$  occur *adiabatically* according to distant observers. This is addition to the adiabatic nature of the perturbing wave as seen by local observers.

A simple calculation shows that small changes of the horizon area are related to those of  $M$  and  $J$  by [24, 3, 25]

$$\Delta A = (\Delta M - \Omega \Delta J) / \Theta_K, \quad (39)$$

where

$$\Theta_K \equiv \frac{1}{2} A^{-1} \sqrt{M^2 - (J/M)^2} \quad (40)$$

is the surface tension [24] or surface gravity [25] of the black hole. The overall changes  $\Delta M$  and  $\Delta J$  must stand in the ratio  $\omega/m$ . This can be worked out from the energy–momentum tensor, but is immediately clear if one thinks of the wave as composed of quanta, each with energy  $\hbar\omega$  and angular momentum  $\hbar m$ , and using conservation energy and angular momentum. Since  $\omega \approx \Omega m$ , it is seen from Eq. (39) that if the black hole is not extremal ( $J \neq M^2$  and so  $\Theta_K \neq 0$ ),  $\Delta A \approx 0$  to the accuracy of the former equality. Therefore, Kerr horizon area is invariant during adiabatic changes of the mass and angular momentum as judged globally. These changes come about from field perturbations which are adiabatic from the perspective of local inertial observers, as stated earlier. It is under these two conditions that the horizon area is an adiabatic invariant.

The last conclusion is, however, inapplicable to the extremal Kerr black hole ( $J = M^2$ ). In this case  $\Theta_K = 0$  so one cannot use Eq. (39) to calculate the change in area, but must work directly with Eq. (35). From Eq. (38) one learns that  $\Omega = 1/2M$  so that  $\Delta J = \Omega^{-1} \Delta M = 2M \Delta M$ . Replacing  $M \rightarrow M + \Delta M$  and  $J \rightarrow J + 2M \Delta M$  in Eq. (35), and subtracting the

original expression gives

$$\Delta A = 8\pi(2 + \sqrt{2})M\Delta M + O((\Delta M)^2). \quad (41)$$

This is not a small quantity; a generic addition of mass  $\Delta M$  will give a  $\Delta A$  of the same order. Thus horizon area of an extremal Kerr hole is not an adiabatic invariant. Sec. 6 gives one more example of the departure of extremal black horizon area from adiabatic invariance.

For nonextremal black holes the conclusion that horizon area is an adiabatic invariant extends to perturbations by electromagnetic waves. One only has to replace the  $Y_{\ell m}(\theta, \varphi)$  in the wave by an electric or magnetic type *vector* spherical harmonic to describe the electromagnetic modes. The conclusion is the same.

## 6. Particle Absorption by Reissner–Nordström Black Hole

Thus far I have illustrated the adiabatic invariant character of black hole horizon area under field–black hole interaction. The present example focuses rather on point particle–black hole interaction. It is none other than the Christodoulou reversible process [26] for a Reissner–Nordström black hole.

Consider a Reissner–Nordström black hole of mass  $M$  and positive charge  $Q$ . The exterior metric is

$$ds^2 = -\chi dt^2 + \chi^{-1} dr^2 + r^2(d\theta^2 + d\varphi^2), \quad (42)$$

with

$$\chi \equiv 1 - 2M/r + Q^2/r^2. \quad (43)$$

At infinity one shoots in radially a classical point particle of mass  $m$  and positive charge  $\varepsilon$  with total relativistic energy adjusted to the value

$$E = \varepsilon Q/r_{\mathcal{H}}. \quad (44)$$

Here  $r_{\mathcal{H}}$  is the  $r$  coordinate of the infinite red shift surface ( $\chi = 0$ ), which by Vishveshwara's theorem [1] coincides with the event horizon:

$$r_{\mathcal{H}} = M + \sqrt{M^2 - Q^2} \quad (45)$$

In Newtonian terms this particle should marginally reach the horizon where its potential energy just exhausts the total energy. The relativistic equation of motion leads to the same conclusion.

For consider the action for the radial motion,

$$S = \int \left[ -m \sqrt{\chi (dt/d\tau)^2 - (dr/d\tau)^2/\chi} - \varepsilon A_t dt/d\tau \right] d\tau, \quad (46)$$

where  $\tau$ , the proper time, acts as a path parameter, and  $A_t = Q/r$  is the only nontrivial component of the electromagnetic 4-potential. The stationary character of the background metric and field means that there exists a conserved quantity, namely

$$E = -\delta S/\delta(dt/d\tau) = \frac{m\chi}{\sqrt{\chi(dt/d\tau)^2 - (dr/d\tau)^2/\chi}} \frac{dt}{d\tau} + \frac{\varepsilon Q}{r}. \quad (47)$$

We also know that the norm of the velocity is conserved. This together with the definition of proper time gives  $\sqrt{\chi(dt/d\tau)^2 - (dr/d\tau)^2/\chi} = 1$ . Substituting  $dt/d\tau$  from here in Eq. (47) gives

$$E = m\sqrt{\chi + (dr/d\tau)^2} + \frac{\varepsilon Q}{r}. \quad (48)$$

It is easy to see that this is precisely the total energy of the particle, for at large distances from the hole,  $E \approx m + mv^2/2 - m M/r + \varepsilon Q/r$  (sum of rest, kinetic, gravitational and electrostatic potential energies). Setting  $E = \varepsilon Q/r_{\mathcal{H}}$  shows that the radial motion has a turning point ( $dr/d\tau = 0$ ) precisely at the horizon [ $\chi(r_{\mathcal{H}}) = 0$ ].

Because the particle's motion has a turning point at the horizon, it gets accreted by it. The area of the horizon is originally

$$A = 4\pi r_{\mathcal{H}}^2 = 4\pi \left( M + \sqrt{M^2 - Q^2} \right)^2, \quad (49)$$

and the (small) change inflicted upon it by the absorption of the particle is

$$\Delta A = (\Delta M - Q \Delta Q/r_{\mathcal{H}})/\Theta_{RN} \quad (50)$$

with

$$\Theta_{RN} \equiv \frac{1}{2} A^{-1} \sqrt{M^2 - Q^2} \quad (51)$$

being the surface gravity analogous to the previous  $\Theta_K$ . Thus if the black hole is not extremal so that  $\Theta_{RN} \neq 0$ ,  $\Delta A = 0$  because  $\Delta M = E = \varepsilon Q/r_{\mathcal{H}}$  while  $\Delta Q = \varepsilon$ . Therefore, the horizon area is invariant under the accretion of the particle from a turning point.

To a momentarily radially stationary local inertial observer, the particle in question hardly moves radially as it is accreted. Thus its assimilation is adiabatic. By contrast, if  $E$  were larger than in (44), the particle would not try to turn around at the horizon, and the local observer would see it moving radially at finite speed and being assimilated quickly. And the horizon's area would increase upon its accretion, as is easy to check from the previous argument. Thus invariance of the area goes hand in hand with adiabatic changes at the horizon as judged by local observers at the horizon.

Needless to say, the changes in  $M$  and  $Q$  also occur very slowly as judged by distant observers. This is the double sense in which the area is an adiabatic invariant.

The above conclusions fail for the extremal Reissner–Nordström black hole. When  $Q = M$ ,  $\sqrt{M^2 - Q^2}$  in Eq. (49) is unchanged to  $O(\varepsilon^2)$  during the absorption, so that  $\Delta A = 8\pi ME$ . This is not a small change, so the horizon’s area is not an adiabatic invariant. It is clear, as already noted earlier, that extremal black holes behave differently from generic black holes in this as in other phenomena.

## 7. Conclusions

The examples here collated suggest the existence of a theorem which would state that, classically, under suitably adiabatic changes (two such are here characterized) of a black hole in equilibrium, the area of its event horizon does not change. Aside from harmonizing with the understanding that horizon area represents entropy, this theorem would provide a formal motivation for quantizing the black hole in the spirit of the “old quantum theory” or Bohr–Sommerfeld quantization. The implications of such quantization have already been considered. [5, 6]

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